Lyapunov stability:

- Consider $V: \mathbb{R}^n \to \mathbb{R}$, $V \in C^1$
- V is p.d if $V(0)=0$ and $V(x)>0$ $\forall x \neq 0$
- V is radially unbounded if

$$V(x) \to \infty \text{ as } \|x\| \to \infty$$

- For the system $\dot{x} = f(x)$, we define

$$\dot{V}(x) = \frac{\partial V}{\partial x}(x)\, f(x)$$

Thm (Thm 4.1 and 4.2 in book)

- Consider $\dot{x} = f(x)$ with $x=0$ as eqlb.
- Let V be p.d.

If

① $\dot{V}(x) \leq 0$ $\forall x \in D$ → open set containing zero

then $x=0$ is stable

② $\dot{V}(x) < 0$ $\forall x \in D - \{0\}$

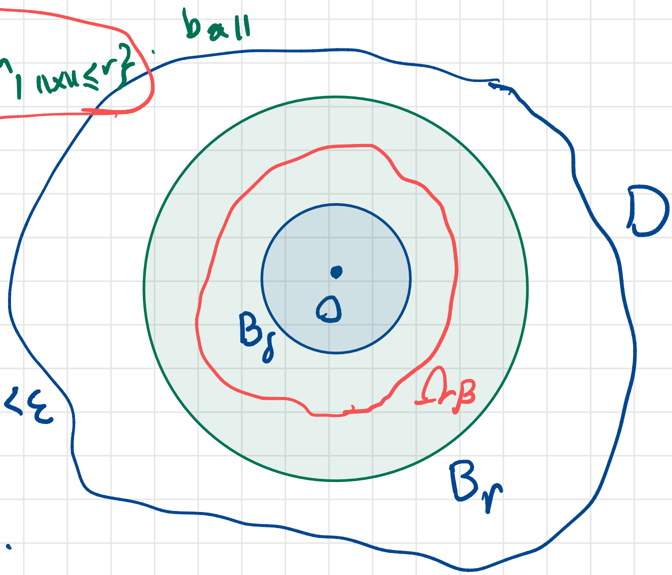
then $x=0$ is AS

③ $\dot{V}(x) < 0$ $\forall x \neq 0$ and V is radially unbounded

$\Rightarrow$ $x=0$ is GAS

proof

Notation: $B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$



(1) To show $\forall \varepsilon > 0$

$\exists \delta$ s.t.

if $\|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon$

- pick $r \in (0, \varepsilon)$ s.t.

$$B_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\} \subset D \rightarrow$$ possible because D is open

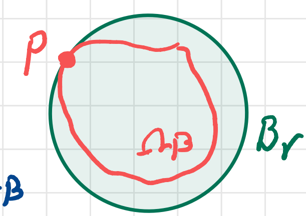
- let $\alpha = \min_{\|x\|=r} V(x)$ and $\beta \in (0, \alpha)$

- let $\Omega_\beta = \{x \in B_r \mid V(x) \leq \beta\}$ level surface

- Then $\Omega_\beta \subset B_r$

If not, then $\exists p \in \Omega_\beta$ s.t. $\|p\| = r$

$\Rightarrow V(p) \geq \alpha > \beta \longrightarrow$ Contradicts $p \in \Omega_\beta$



- If $x(0) \in \Omega_\beta$, then $x(t) \in \Omega_\beta$ because

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta \quad \forall t \geq 0$$

- we need to chose δ small enough s.t. $B_\delta \subset \Omega_\beta$

- Because V is continuous at $x=0$, then

$\exists\, \delta > 0$ s.t. if $\|x - 0\| \leq \delta$ $\rightarrow$ $\|x\| \leq \delta$

then $|V(x) - \underbrace{V(0)}_{0}| < \beta$ $\rightarrow$ $V(x) < \beta$

$\Rightarrow B_\delta \subset \Omega_\beta \subset B_r$

$\Rightarrow$ if $\underbrace{\|x(0)\| \leq \delta}_{x_0 \in B_\delta}$ $\xRightarrow{V \text{ is cont.}}$ $x(0) \in \Omega_\beta$

$\xRightarrow{\dot{V} \leq 0}$ $x(t) \in \Omega_\beta$

$\xRightarrow{\text{by def of } \Omega_\beta}$ $x(t) \in B_r$

$\xRightarrow{\text{by def of } B_r}$ $\|x(t)\| \leq r < \varepsilon$ ∎

② Proof for AS

- Now, assume $\dot{V}(x) < 0$ $\forall x \in D$, $x \neq 0$

- Need to show $x(t) \rightarrow 0$ as $t \rightarrow \infty$

- It is sufficient to show $V(x(t)) \rightarrow 0$

because $V(x(t)) \rightarrow 0$ happens if $x(t) \rightarrow 0$

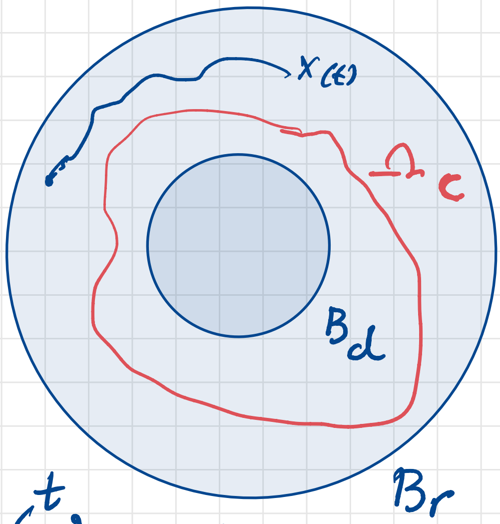
or $x(t) \rightarrow \infty$, but from stability, $x(t)$ is bdd.

- $\dot{V}(x(t)) < 0$ $\Rightarrow$ $V(x(t))$ is monotonically decreasing

$V(x(t)) \geq 0$ and bdd from below.

$\Rightarrow$ $V(x(t))$ converges to some number c

- If $c = 0$, we are done
- To show $c = 0$, we use contradiction
- Suppose $c > 0$. Choose d small enough s.t. $B_d \subset \Omega_c$



- Because $V(x(t)) \geq c$

  Then, $\|x(t)\| \geq d$
- let $-\gamma = \max\limits_{d \leq \|x\| \leq r} \dot{V}(x)$
- By assumption ($\dot{V} < 0$) $\gamma > 0$
- $$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))\, ds$$

  $$\leq V(x(0)) - \gamma t$$
- the RHS becomes negative and contradicts $V(x(t)) \geq c$

$\blacksquare$

③ Proof for GAS

- Need to show $x(t) \longrightarrow 0$ as $t \to \infty$ from any initial condition (I.C.)

- It is enough to show that $x(t)$ is bounded starting from any I.C. or $\|x(t)\| \leq r$ for some r

- Because, then, we can use the same arguments as in part ② for AS to show $x(t) \longrightarrow 0$

- To show $x(t)$ is bounded, we use the fact that V is radially unbounded
- For any initial condition, let $c = V(x(0))$

- $V(x) \to \infty$ as $\|x\| \to \infty$ implies that
  $\exists r > 0$ s.t. $V(x) > c$ for all $\|x\| > r$

- Therefore, $\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \leq c\} \subset B_r$

- $x(0) \in \Omega_c \overset{\dot{V} < 0}{\Longrightarrow} x(t) \in \Omega_c \Rightarrow x(t) \in B_r$
  $\Rightarrow x(t)$ is bounded ∎

Remark:

- function V satisfying p.d and $\dot{V} \leq 0$ is called Lyapunov function
- Finding the Lyapunov function is art and based on experience and intuition $\Leftrightarrow$ No principled way

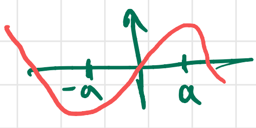
Examples: (scalar systems)

- Consider $\dot{x} = -g(x)$ $\quad x \in \mathbb{R}$

  where $\underline{g(x)\,x > 0}$ $\forall\, x \neq 0$ and $|x| \leq a$

  and $x = 0$ is eqlb. point

$$\begin{cases} g(x) > 0 & \text{if } x \in (0, a) \\ g(x) < 0 & \text{if } x \in (-a, 0) \\ g(0) = 0 & \end{cases}$$



for example $g(x) = x$ or x^3

- Pictorially, we can see $x = 0$ is AS

  since $x(t) > 0$ implies $\dot{x} = -g(x) < 0$

  $x(t) < 0$ implies $\dot{x} = -g(x) > 0$

- We can conclude AS using Lyapunov funct.
- Define $V(x) = \int_0^x g(y)\,dy$
- Then, V is p.d. because $V(0) = 0$
$V(x) > 0$
- Also, $\dot{V}(x) = -\frac{\partial V}{\partial x}(x)\, g(x)$
$= -g^2(x) < 0 \quad \forall x \neq 0$

$\Rightarrow x = 0$ is AS.

<u>Example:</u> (Energy function)

- Consider Pendelum $\dot{x}_1 = x_2$
$\dot{x}_2 = -a\sin(x_1) - bx_2$
- Energy is usually a good candidate for Lyp func.
- $V(x) = \frac{1}{2}x_2^2 + a(1-\cos(x_1))$

$\Rightarrow V(0) = 0, \quad V(x) > 0 \quad \forall x \neq 0$

$\Rightarrow \dot{V}(x) = \frac{\partial V}{\partial x} f = [a\sin(x_1),\ x_2] \begin{bmatrix} x_2 \\ -a\sin(x_1) - bx_2 \end{bmatrix}$

$$\Rightarrow \dot{V}(x) = a\, x_2 \sin(x_1) - a\, x_2 \sin(x_1) - b x_2^2$$
$$= -b x_2^2$$

- If $b=0 \Rightarrow \dot{V} \leq 0 \Rightarrow x=0$ is stable and not AS since $\dot{V}(x)=0$
- If $b>0$, we can not apply the theorem because $\dot{V}(x)=0$ if $x_2=0$ but x_1 is arbitrary
- But we know that $x=0$ is AS.
- Later, we introduce Lasalle thm for this case
- But, now let's try modifying the Lyapunov func
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$$V(x) = \frac{1}{2} x^T P x + a(1-\cos(x_1))$$

$$= \frac{1}{2} [x_1, x_2] \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1-\cos(x_1))$$

- $V(x)$ is p.d if matrix P is p.d.
- This is true when $\underbrace{P_{11} > 0, \quad P_{11}P_{22} - P_{12}^2 > 0}_{\text{principal minors.}}$

- $\dot{V}(x) = \frac{\partial V}{\partial x} f$

$$\frac{\partial V}{\partial x} = [P_{11}x_1 + P_{12}x_2 + a\sin(x_1),\ P_{22}x_2 + P_{12}x_1]$$

$$\Rightarrow \dot{V} = P_{11}x_1x_2 + P_{12}x_2^2 + a x_2 \sin(x_1)$$
$$- P_{22}x_2 a\sin(x_1) - P_{12}x_1 a\sin(x_1)$$
$$- bP_{22}x_2^2 - bP_{12}x_1x_2$$

$$= (P_{12} - bP_{22})x_2^2 + (P_{11} - bP_{12})x_1x_2$$
$$+ (1 - P_{22})a x_2 \sin(x_1) - P_{12} a x_1 \sin(x_1)$$

- we want $\dot{V}(x) < 0 \Rightarrow$ $P_{11} = bP_{12}$, $P_{22} = 1$ to remove the cross terms

and

$bP_{22} > P_{12}$ to make it negative

- for p.d. we have $P_{12}^2 \leq P_{11}P_{22} = P_{11} = bP_{12}$

$\Rightarrow P_{12} \leq b$ and $P_{12} > 0$

- Take $P_{12} = \frac{b}{2} \rightarrow P_{11} = \frac{b^2}{2}$, $P_{22} = 1$

$$\Rightarrow \dot{V} = -\frac{b}{2}x_2^2 - \frac{ab}{2}x_1\sin(x_1) \Rightarrow \dot{V} < 0$$